Introduction to homological alg.

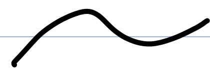
Ref. Ch. A. Weibel. An introduction to homological alg.

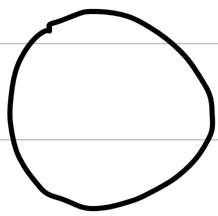
F. H. Croom. Basic concepts of algebraic topology

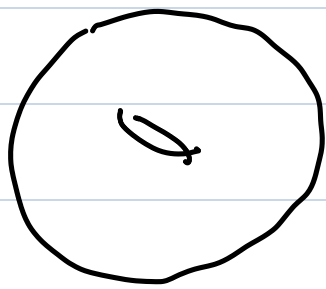
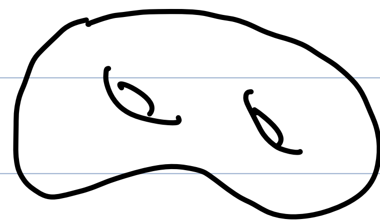
D. Eisenbud. Commutative algebra with a view toward algebraic geometry.

§ Motivation from alg. top.

X : top'l sp. ← set + topology

<u>Examples</u> (1) · (2) 

(3)  (4) 

(5)  (6) 

(7)  (8) $\mathbb{R}^n$

(9) $\mathbb{CP}^n$ etc ...

§ Category thy

<u>Def</u> A category $\mathcal{C}$ consists of the following:

- a class $Obj(\mathcal{C})$ of objects
- a set $Hom_{\mathcal{C}}(A,B)$ of morphisms for every ordered pair (A,B) of objs
- an identity morphism $id_A \in Hom_{\mathcal{C}}(A,A)$ for each obj A
- a composition fn

$$Hom_{\mathcal{C}}(A,B) \times Hom_{\mathcal{C}}(B,C) \to Hom_{\mathcal{C}}(A,C)$$

for every ordered triple (A,B,C)

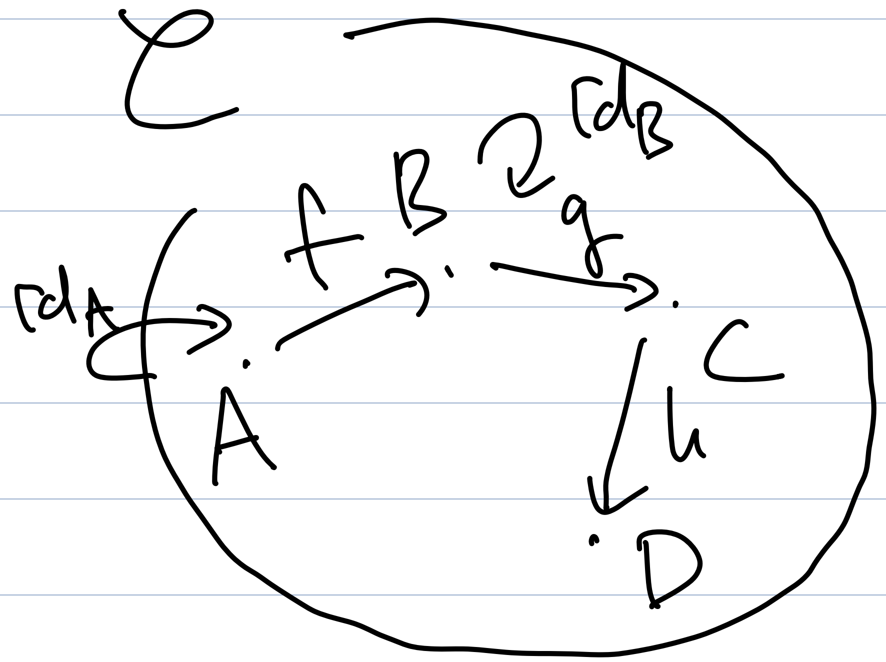
of objs.

⌜We write $f: A \to B$ to indicate that f is a morphism in $\mathrm{Hom}_{\mathcal{C}}(A, B)$ & we write gf or $g \circ f$ for the composition of $f: A \to B$ with $g: B \to C$.⌟

- The above data is subject to the following axioms.

Associativity axiom: $(hg)f = h(gf)$, $\forall f: A \to B$, $\forall g: B \to C$, $\forall h: C \to D$

Unit axiom: $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$ $\forall f: A \to B$



Example (1) Set : the cat. of sets.

Objs : sets

Morphisms : ftns.

Composition : Composition of ftns

(2) Ab : the cat. of abelian gps.

Objs : abelian gps

Morphisms : gp. homos

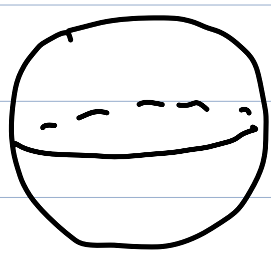
Composition : Composition of homos.

§ Topological invariants

<u>Def</u> (Category of top'l sps) Top.

Objs : top'l sp.
Morphisms : conti. maps
Composition : Composition of conti. maps

<u>Question</u> How to distinguish two top'l sps?

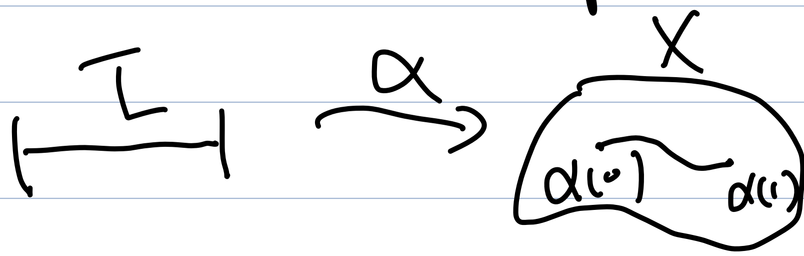
 $\not\simeq$ 

<u>Idea.</u>  $\not\simeq$ 

§ Fundamental gps.

Def (1) A path in a top'l sp. X is a conti. map $\alpha : I \longrightarrow X$.

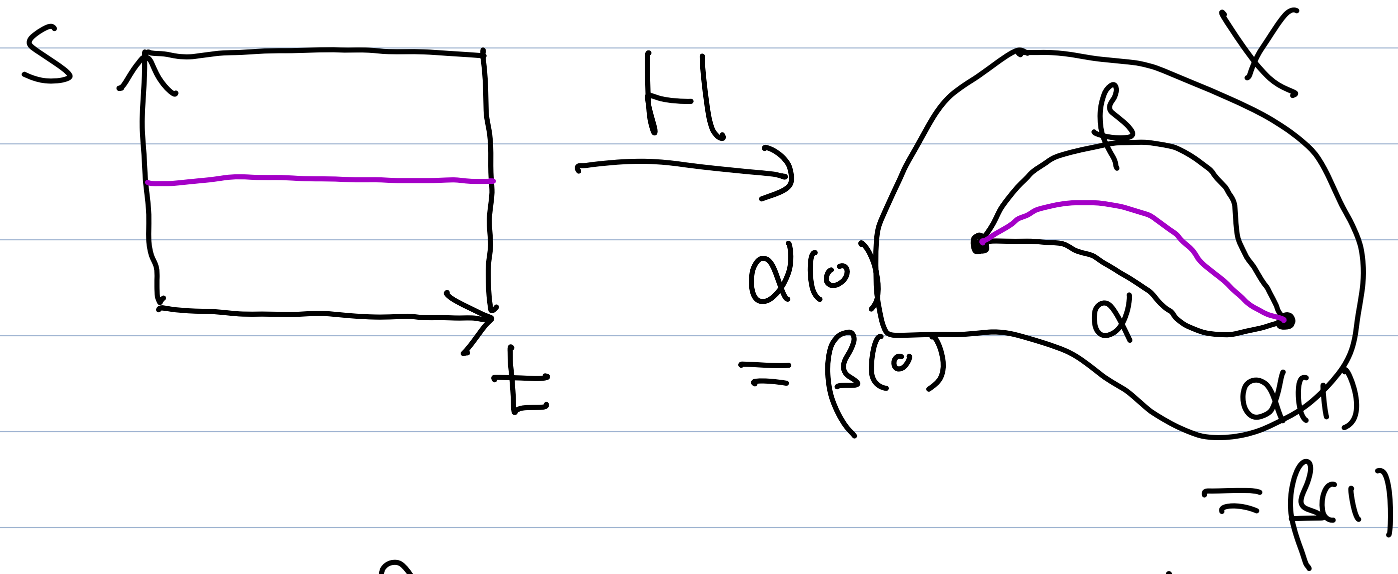
$I = [0,1]$



(2) Paths α & β with common initial pt $\alpha(0) = \beta(0)$ & terminal pt $\alpha(1) = \beta(1)$ are equiv. if $\exists$ a conti. map $H : I \times I \to X$ s.t.

$$\begin{cases} H(t,0) = \alpha(t), & t \in I \\ H(t,1) = \beta(t), & t \in I \end{cases}$$

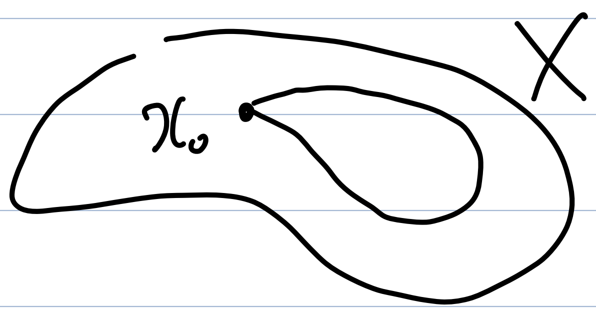
$$\begin{cases} H(0,s) = \alpha(0) = \beta(0), & s \in I \\ H(1,s) = \alpha(1) = \beta(1), & s \in I \end{cases}$$



(3) The ftn H is called a **homotopy** between α & β.

Def (1) A **loop** in a top'l sp. X is a path α in X with $\underbrace{\alpha(0)=\alpha(1)}_{\text{base pt.}}$.

(2) Two loops α & β having common base pts are **equiv.** if they are equiv. as paths.



Def If α & β are paths in X with $\alpha(1)=\beta(0)$, then the path product $\alpha * \beta$ is the path defined by

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Def (Fundamental gp)

X : top'l sp., $x_0 \in X$

$$\pi_1(X, x_0) = \left\{ \begin{array}{l} \text{loops in } X \\ \text{with base pt} \end{array} \right\} \Big/ \sim_{\text{htpy}}$$

$\alpha \sim \beta$ if α is equiv. to β as loops

Thm The set $\pi_1(X, x_0)$ is a gp. under $\circ$ operation. $[\alpha] \cdot [\beta] := [\alpha * \beta]$

§ Functor

<u>Def</u> $\mathcal{C}, \mathcal{D}$: categories.

A <u>functor</u> $F : \mathcal{C} \to \mathcal{D}$ is a rule that associates an obj. $F(c)$ of $\mathcal{D}$ to every obj c of $\mathcal{C}$ & a morphism $F(f) : F(c_1) \to F(c_2)$ in $\mathcal{D}$ to every morphism $f : c_1 \to c_2$ in $\mathcal{C}$. We require F to preserve identity morphisms ($F(id_c) = id_c$, $\forall c \in \mathrm{Obj}\,\mathcal{C}$) & composition ($F(g \circ f) = F(g) \circ F(f)$).

$$(X, x_0) \longmapsto \pi_1(X, x_0)$$

<u>Example</u> $\pi_1 : \left\{ \begin{array}{c} \text{top'l sps} \\ \text{with base pts} \end{array} \right\} \longrightarrow \text{Groups}.$

§ Homology

<u>Rmk</u> $\pi_1(X, x_0)$ is a non-abelian gp. & is hard to compute in general.

Want a functor (or an inv.) which is each to compute.

H_n : top'l sp. $\longrightarrow$ Ab.

H^* : top'l sp $\longrightarrow$ graded R-alg.

<u>Def</u> (1) A set $A = \{a_0, \cdots, a_k\}$ of $k+1$ pts in $\mathbb{R}^n$ is geometrically indep. if no hyperplane of dim $k-1$ contains all of the pts.

(2) $\{a_0, \cdots, a_k\}$: geometrically indep. pts in $\mathbb{R}^n$. The k-dim'l geometric simplex or k-simplex σ^k spanned by $\{a_0, \cdots, a_k\}$ is

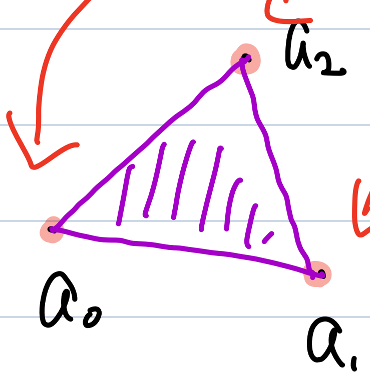
barycentric coordinates of x

$$\sigma^k = \{x \in \mathbb{R}^n \mid x = \sum_{i=0}^{k} \lambda_i a_i,\ \lambda_i \geq 0,\ \sum_{i=0}^{k} \lambda_i = 1\}.$$

$$\| $$

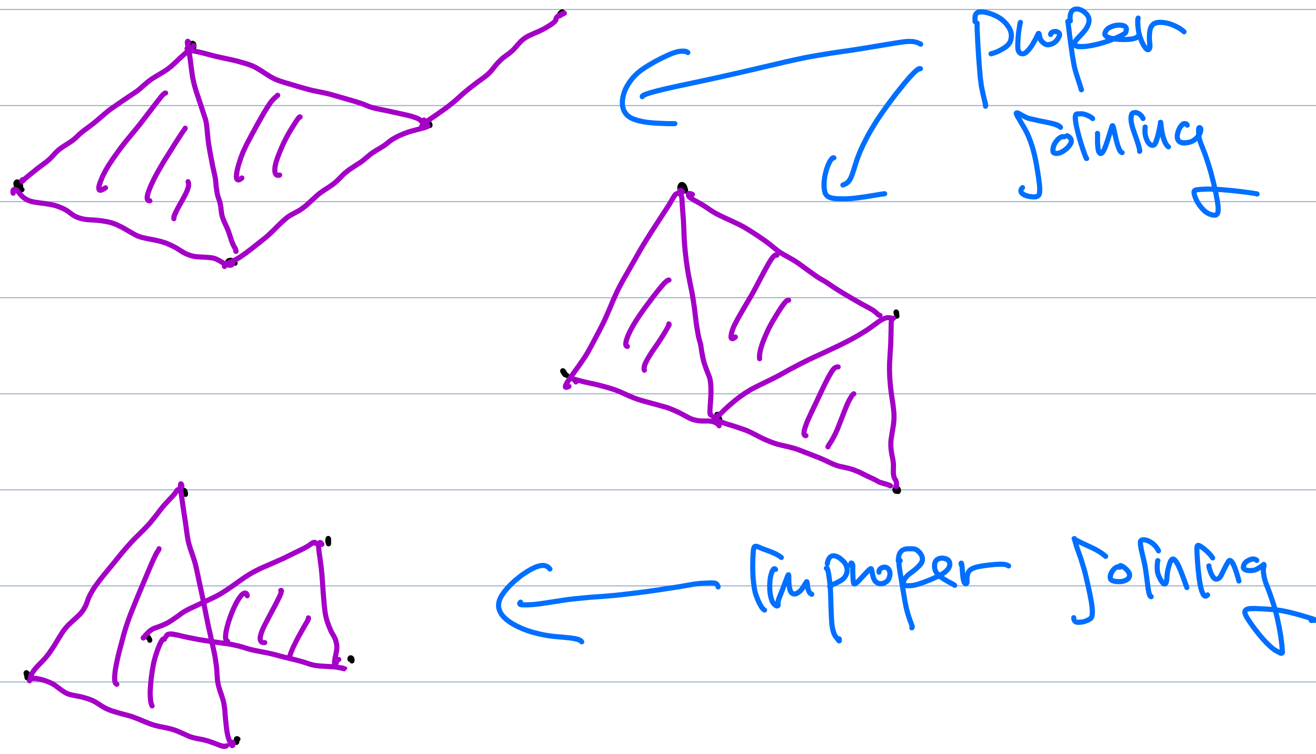
$$\langle a_0, \cdots, a_k \rangle$$

vertices of σ^k



(3) A simplex σ^k is a face of a simplex σ^n, $k \leq n$ if each vertex of σ^k is a vertex of σ^n.

(4) Two simplexes σ^m & σ^n are properly joined if they do not intersect or the intersection $\sigma^m \cap \sigma^n$ is a face of both σ^m & σ^n.



Def A geometric cpx. (or simplicial cpx) is a finite family K of geometric simple-xes which are properly joined & have the property that each face of a member of K is also a member of K. The dimension of K is the largest positive integer n s.t. K has an n-simplex.

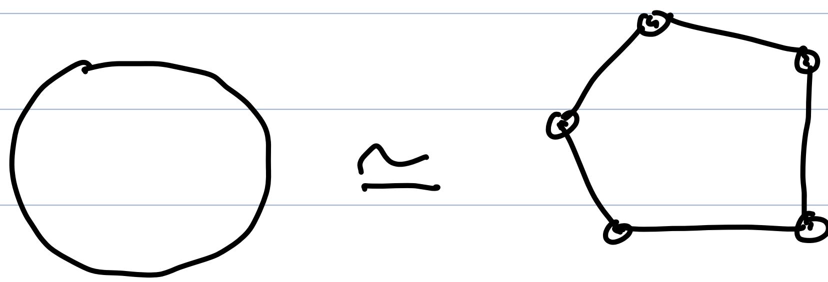
The union of members of K with the Euclidean subsp. top. is denoted by $|K|$ &

is called the polyhedron assoc. with K.

Def X : top'l sp.

If there is a geometric cpx. K with $|K| \underset{\text{homeo}}{\simeq} X$, then X is said to be triangulable sp & the cpx. K is called a triangulation of X.

Example (1)



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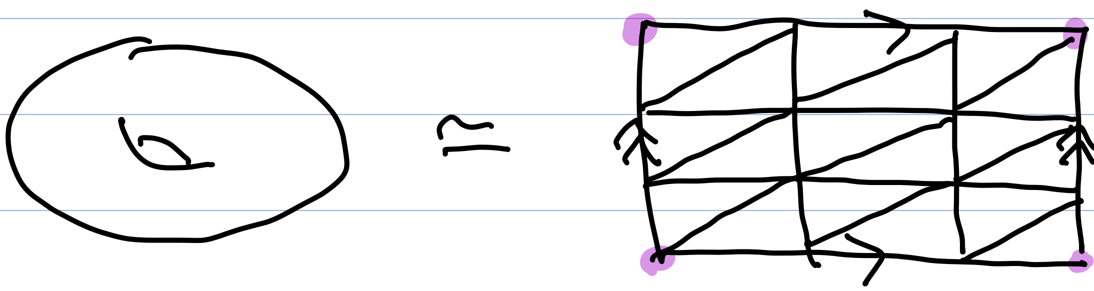

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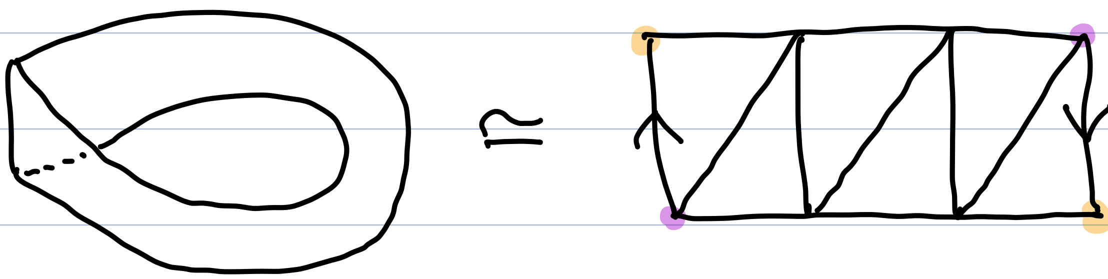
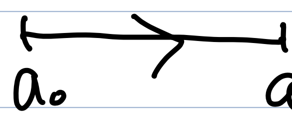


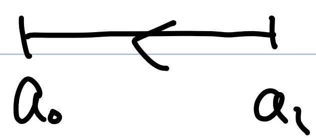
Def An oriented n-simplex, $n \geq 1$, is obtained from an n-simplex $\sigma^n = \langle a_0, \cdots, a_n \rangle$ by choosing an ordering for its vertices. The equiv. class of even odd permutations of the chosen ordering determines the positively oriented simplex $+\sigma^n$ $(-\sigma^n)$

An oriented geometric cpx. is obtained from a geometric cpx by assigning an orientation to each of its simplexes.

Example (1) $\sigma^1 = \langle a_0, a_1 \rangle$

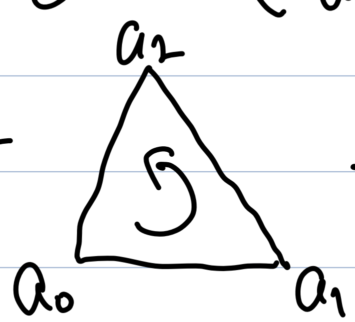


$\Rightarrow -\sigma^1 = \langle a_1, a_0 \rangle$

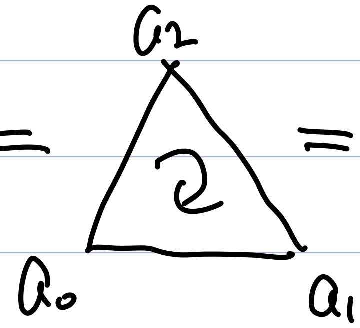


(2) $\sigma^2 = \langle a_0, a_1, a_2 \rangle$

$+\sigma^2$



$\Rightarrow -\sigma^2 =$



$= \langle a_0, a_2, a_1 \rangle$

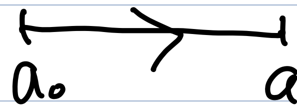
Def K : oriented simplicial cpx.

(1) $C_p(K) := \sum \langle \sigma^p \rangle_{\sigma^p : \text{p-dim'l simplex of } K.}$

(2) $\partial : C_p(K) \longrightarrow C_{p-1}(K)$

$$\sigma^p = \langle a_0, \cdots, a_p \rangle \longmapsto \sum_{i=0}^{p} (-1)^i \langle a_0, \cdots, \hat{a_i}, \cdots, a_p \rangle$$

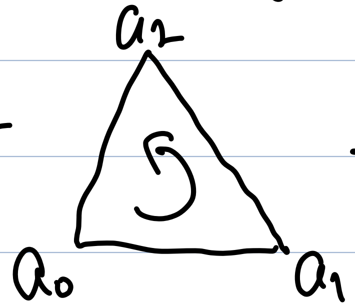
Example (1) $\sigma^1 = \langle a_0, a_1 \rangle$



$\Rightarrow \partial \sigma^1 = \langle a_0 \rangle - \langle a_1 \rangle$

(2) $\sigma^2 = \langle a_0, a_1, a_2 \rangle$

$+\sigma^2$



$\Rightarrow \partial \sigma^2 = \langle a_0, a_1 \rangle - \langle a_0, a_2 \rangle + \langle a_1, a_2 \rangle$

$\Rightarrow \longrightarrow - \nearrow + \nwarrow$

Rmk $\partial(\partial \sigma^2) = a_0 - a_1 - a_0 + a_2 + a_1 - a_2 = 0.$

Thm If K is an oriented cpx, then

$\partial^2 : C_p(K) \longrightarrow C_{p-2}(K)$ is 0.

Def K is an oriented cpx

(1) $Z_p(K) := \ker(\partial : C_p(K) \longrightarrow C_{p-1}(K))$

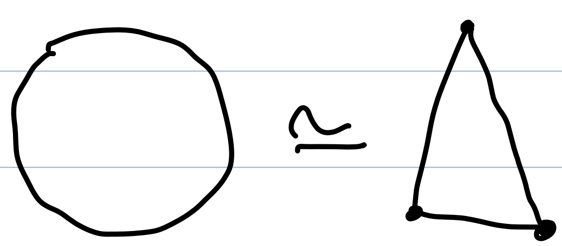
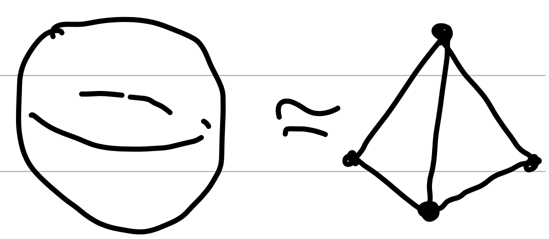
$$C_p(K) \ni \underbrace{z}_{p\text{-cycle}} \longmapsto 0 \in C_{p-1}(K)$$

(2) $B_p(K) := \operatorname{Im}(\partial : C_{p+1}(K) \longrightarrow C_p(K))$

$$C_p(K) \ni \underbrace{z}_{p\text{-bdry}}$$

(3) $\underbrace{H_p(K)}_{p\text{-dimensional homology of } K.} := Z_p(K)/B_p(K)$

Examples (1) 

(2)  (3) 

<u>Thm</u> H_p : nice top'l sp $\longrightarrow$ Ab

e.g. triangulable sps

is a functor.

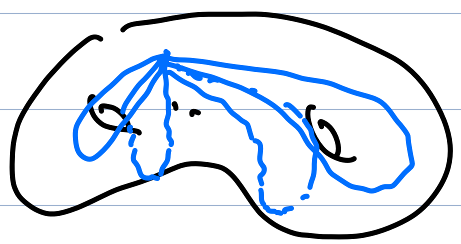
<u>Examples</u> (1) $H_p(\vdash\!\dashv, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p=0 \\ 0 & \text{o.w.} \end{cases}$

(2) $H_p(S^1, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p=0,1 \\ 0 & \text{o.w.} \end{cases}$

(3) $H_p(S^2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p=0,2 \\ 0 & \text{o.w.} \end{cases}$

(4) $H_p(T^2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p=0,2 \\ \mathbb{Z}^2 & \text{if } p=1 \\ 0 & \text{o.w.} \end{cases}$

(5) $H_p(\Sigma_g, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p=0,2 \\ \mathbb{Z}^2 & \text{if } p=1 \\ 0 & \text{o.w.} \end{cases}$



g holes

(6) $H_p(S^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p = 0, n \\ 0 & \text{o.w.} \end{cases}$

...

<u>Thm</u> X : nice top'l sp.

(1) $H_0(X, \mathbb{Z}) = \mathbb{Z}^{\#\text{ of cnt. comps of } X}$

(2) If $H_1(X, \mathbb{Z}) = \pi_1(X, x_0)^{ab}$.

...

§ Modules

Def R : ring

A left R-mod. M is an abelian gp. with an action of R, that is, a map

$$R \times M \longrightarrow M \quad \text{satisfying}$$
$$(r, m) \longmapsto rm$$

- Associativity : $r(sm) = (rs)m$
- Distributivity : $r(m+n) = rm + rn$

$$(r+s)m = rm + sm$$

$$1 \cdot m = m$$

$\forall r, s \in R$, $\forall m, n \in M$.

Example $Ab = \mathbb{Z}$-mods

$V.S./k = k$-mods (k : field)

Def M, N : left R-mod

(1) A map $f: M \to N$ is an R-mod. homo. if it is an abelian gp. homo. & $f(rm) = rf(m)$, $\forall r \in R, m \in M$.

(2) An abelian subgp. $L \leq M$ is an R-sub-mod if $rl \in L$, $\forall r \in R, l \in M$. When $L \leq M$ is a submod, then M/L becomes an R-mod.

$$\ulcorner L \underset{\text{submod}}{\leq} M \iff i: L \xrightarrow{\text{inj R-mod homo.}} M \lrcorner$$

(3) $f: M \to N$ R-mod. homo.

$\Rightarrow \ker f := \{m \in M \mid f(m) = 0\} \leq M$

& $\operatorname{Im} f := \{f(m) \mid m \in M\} \leq N$

& $\operatorname{coker} f := N/\operatorname{Im} f$.

§ Additive categories

Def (1) A cat. $\mathcal{A}$ is called an **Ab-cat.** if every hom-set $\mathrm{Hom}_{\mathcal{A}}(A, B)$ in $\mathcal{A}$ is given the str. of abelian gp in such a way that composition distributes over addition.

(2) A functor $F: \mathcal{A} \to \mathcal{B}$ is **additive** if $\mathrm{Hom}_{\mathcal{A}}(a, a') \to \mathrm{Hom}_{\mathcal{B}}(Fa, Fa')$ is a gp. homo.

(3) An **additive cat.** is an Ab-cat. $\mathcal{A}$ with a zero obj (an obj that is initial & terminal) & a product $A \times B$ for every pair A, B of objs in $\mathcal{A}$.

Example R-mod : the cat. of R-mods is an additive cat.

Def $\mathcal{A}$: additive cat.

(1) A kernel of a morphism $f: M \to N$ is a morphism $i: \ker f \to M$ s.t. $f \circ i = 0$ & that is univ. w.r.t. this pro-perty.

(2) A cokernel of a morphism $f: M \to N$ is a morphism $\pi: N \to \operatorname{coker} f$ s.t. $\pi \circ f = 0$ & that is univ. w.r.t. this pro-perty.

(3) A morphism $j: A \to B$ is monic if $j \circ g = 0 \Rightarrow g = 0$ for every $g: A' \to A$.

(4) A morphism $q: C \to D$ is epi if $h \circ q = 0 \Rightarrow h = 0$ for every $h: D \to D'$

§ Abelian categories

Def An abelian cat. is an additive cat. $\mathcal{A}$ s.t.

1. every map in $\mathcal{A}$ has a kernel & cokernel.
2. every monic in $\mathcal{A}$ is the kernel of its cokernel
3. every epi in $\mathcal{A}$ is the cokernel of its kernel.

Thm R-mod is an abelian cat.

Def A cat. is small if its class of objs is infact a set.

$\mathrm{Hom}_{\mathcal{A}}(M,N) \cong \mathrm{Hom}_R(M,N)$

Thm (Freyd–Mitchell embedding thm)
If $\mathcal{A}$ is a small abelian cat, then $\exists$ a ring R & an ext. fully faithful functor $\mathcal{A} \to R\text{-mod}$ which embeds $\mathcal{A}$ as a full subcat.

Lemma $\mathcal{C} \underset{\text{full}}{\subset} \mathcal{A}$, $\mathcal{A}$: abelian cat.

1. $\mathcal{C}$ is additive
$\iff 0 \in \mathcal{C}$ & $\mathcal{C}$ is closed under $\oplus$

2. $\mathcal{C}$ is abelian & $\mathcal{C} \subset \mathcal{A}$ is ext
$\iff \mathcal{C}$ is additive & $\mathcal{C}$ is closed under ker & coker.

additive cat.

Example $\left[\begin{array}{l}\text{Cat. of free} \\ \text{abelian gps}\end{array}\right] \subset Ab$ the 1st isom. thm.

Rmk The conditions 2, 3 can be replaced by

$$\ker f \xrightarrow{i} M \xrightarrow{f} N \xrightarrow{\pi} \operatorname{Coker} f$$

the 1st isom. thm.

$$\operatorname{Coker} i = \operatorname{CoIm} f \cong \operatorname{Im} f = \ker \pi$$

§ Complexes of R-modules

<u>Def</u> A chain cpx $C_\cdot$ of R-mods is a family $\{C_n\}_{n\in\mathbb{Z}}$ of R-mods, together with R-mod maps $d = d_n : C_n \to C_{n-1}$ s.t. $d^2 : C_n \to C_{n-2}$ is zero, $\forall n \in \mathbb{Z}$.

differential

$Z_n(C_\cdot) = \ker(d_n : C_n \to C_{n-1})$

the mod. of n-cycles

$B_n(C_\cdot) = \operatorname{Im}(d_{n+1} : C_{n+1} \to C_n)$

the mod. of n-boundaries

$\Rightarrow 0 \subseteq B_n \subseteq Z_n \subseteq C_n$

$H_n(C.) := Z_n(C.) / B_n(C.)$

the nth homology of $C.$

Def $Ch(\text{mod-}R)$: the cat. of chain cpxes of (right) R-mods

Objects : chain cpxes of R-mods.

Morphism : $\mathrm{Hom}_{Ch(R)}(C.\, D.) = \{ u : C. \to D. \mid u_n : C_n \to D_n$ R-mod. homo.

s.t.
$$\begin{tikzcd} C_n \arrow[r, "d"] \arrow[d, "u_n"'] \arrow[dr, phantom, "\circlearrowleft"] & C_{n-1} \arrow[d, "u_{n-1}"] \\ D_n \arrow[r, "d"'] & D_{n-1} \end{tikzcd}$$
$\forall n \in \mathbb{Z} \}$.

$u \in \mathrm{Hom}_{Ch(R)}(C.\, D.)$ is called a chain map.

Def $\mathcal{A}$: abelian cat.

$Ch(\mathcal{A})$: the cat. of chain complexes of objs in $\mathcal{A}$

Objs $\{ \xrightarrow{d} C_{p+1} \xrightarrow{d} C_p \xrightarrow{d} C_{p-1} \xrightarrow{d} \cdots \}$

s.t. $d^2 = 0$.

Morphisms $\mathrm{Hom}_{Ch(\mathcal{A})} = \{ \underbrace{f : C_\cdot \to D_\cdot}_{\text{chain maps}} \}$

Thm $Ch(\mathcal{A})$ is an abelian cat.

Exercise $f \in \mathrm{Hom}_{Ch(\mathcal{A})}(C_\cdot, D_\cdot)$

Induces $H_n(f) : H_n(C_\cdot) \to H_n(D_\cdot)$

& $H_n : Ch(\mathcal{A}) \to \mathcal{A}$ is a functor, $\forall n \in \mathbb{Z}$.

§ Long exact sequences

<u>Lemma</u> (Snake lemma)

Consider a comm. diagram of R-mods of the form

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "f"'] & B' \arrow[r, "p"] \arrow[d, "g"'] & C' \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r] & C
\end{tikzcd}$$

If the rows are ext, then $\exists$ an ext. seq. $\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\partial} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h)$

with $\partial(c') = i^{-1} g p^{-1}(c')$, $c' \in \ker(h)$. Moreover, if $A' \to B'$ is monic, then so is $\ker(f) \to \ker(g)$ & if $B \to C$ is epi, then so is $\operatorname{coker}(g) \to \operatorname{coker}(h)$.

<u>Thm</u> Let $0 \to A_\cdot \xrightarrow{f} B_\cdot \xrightarrow{g} C_\cdot \to 0$ be a s.e.s. of chain cpxes. Then $\exists$ natural maps $\partial : H_n(C_\cdot) \to H_{n-1}(A_\cdot)$ called connecting homo. s.t.

$$\cdots \xrightarrow{g} H_{n+1}(C_\cdot) \xrightarrow{\partial} H_n(A) \xrightarrow{f} H_n(B_\cdot) \xrightarrow{g}$$

is an ext. seq.

Sketch of pf) Construction of ∂

From the snake lemma & the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z_nA \arrow[r] \arrow[d] & Z_nB \arrow[r] \arrow[d] & Z_nC \arrow[d] & \\
0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

ext.

$$\frac{A_{n-1}}{dA_n} \longrightarrow \frac{B_{n-1}}{dB_n} \longrightarrow \frac{C_{n-1}}{dC_n} \longrightarrow 0$$

ext.

$$\Longrightarrow \quad \begin{tikzcd}
 & H_n(A) \arrow[r] \arrow[d] & H_n(B) \arrow[r] \arrow[d] & H_n(C) \arrow[d] \arrow[ddddlll, bend left, "\partial"] & \\
 & \frac{A_n}{dA_{n+1}} \arrow[r] \arrow[d, "d"'] & \frac{B_n}{dB_{n+1}} \arrow[r] \arrow[d, "d"'] & \frac{C_n}{dC_{n+1}} \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r] \arrow[d] & Z_{n-1}(B) \arrow[r] \arrow[d] & Z_{n-1}(C) \arrow[d] & \\
 & H_{n-1}(A) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(C) &
\end{tikzcd}$$

ext

& the l.e.s. is obtained by pasting these sequences together. $\boxed{h}$

§ Chain homotopies

<u>Def</u> We say that two chain maps $f, g: C_\cdot \to D_\cdot$ are chain homotopic if $f - g = sd + ds$.

<u>Lemma</u> If f & g are chain homotopic, then $H_n(f) = H_n(g): H_n(C_\cdot) \to H_n(D_\cdot)$

<u>Def</u> (Homotopy cat. $K(A)$)

$K(A)$

chain htpy

$\text{Objs } K(A) = \text{Objs } Ch(A)$

Morphisms $\text{Hom}_{K(A)}(C_\cdot, D_\cdot) = \text{Hom}_{Ch(A)}(C_\cdot, D_\cdot)/\sim$

<u>Rmk</u> $K(A)$ is not an abelian cat. in general, but it becomes a triangulated cat.

§ Mapping Cones & Cylinders

Def $f : B_\cdot \to C_\cdot$ a chain map.
The mapping cone of f is the chain cpx. $\mathrm{cone}(f)$ whose deg n part is $B_{n-1} \oplus C_n$.
The differential is given by the cpx.

$$\begin{bmatrix} -d_B & 0 \\ -f & d_C \end{bmatrix} : \begin{matrix} B_{n-1} \\ \oplus \\ C_n \end{matrix} \longrightarrow \begin{matrix} B_{n-2} \\ \oplus \\ C_{n-1} \end{matrix}$$

Check $\begin{bmatrix} -d_B & 0 \\ -f & d_C \end{bmatrix}\begin{bmatrix} -d_B & 0 \\ -f & d_C \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$B[p]_n = B_{n+p}$ with $(-1)^p d$

Rmk $\exists$ a s.e.s.

$$0 \to C \to \mathrm{Cone}(f) \xrightarrow{\delta} B[-1] \to 0$$

$$c \longmapsto (0, c)$$

$$(b, c) \longmapsto -b$$

$$0 \to C_{n+1} \to B_n \oplus C_{n+1} \to B_n \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & C_n \arrow[r] \arrow[d] & B_{n-1}\oplus C_n \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r] \arrow[d] & B_{n-2}\oplus C_{n-1} \arrow[r] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

$$(b,c) \mapsto -b, \qquad \mapsto db$$

$$\begin{bmatrix} -d_B & 0 \\ -f & d_C \end{bmatrix}\begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} -db \\ dc - fb \end{bmatrix}$$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & B_n\oplus C_{n+1} \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n \arrow[r] \arrow[d] & B_{n-1}\oplus C_n \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r] \arrow[d] & B_{n-2}\oplus C_{n-1} \arrow[r] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

$$c \mapsto (0,c)$$

$$dc \mapsto (0,dc)$$

$\Downarrow$ homology l.e.s.

$$\begin{array}{c} H_n(B) \\ \cong \end{array}$$

$$\cdots \to H_{n+1}(C_\cdot) \to H_{n+1}(\mathrm{Cone}(f)) \xrightarrow{\delta_*} H_{n+1}(B[-1])$$

$$\xrightarrow{\partial} H_n(C) \to H_n(\mathrm{Cone}(f)) \to H_{n-1}(B) \to \cdots$$

Lemma The map ∂ is f_*

pf) If $b \in B_n$ is a cycle, the elt. $(-b, 0) \in \mathrm{Cone}(f)_{n+1}$ lifts b via δ.

$$\Rightarrow \begin{bmatrix} -d_B & 0 \\ -f & d_C \end{bmatrix} \begin{bmatrix} -b \\ 0 \end{bmatrix} = \begin{bmatrix} db \\ fb \end{bmatrix} = \begin{bmatrix} 0 \\ fb \end{bmatrix}$$

$$\Rightarrow \partial[b] = [fb] = f_*[b].$$ $\square$

Def (Mapping cylinder)

$f : B_\cdot \longrightarrow C_\cdot$ chain cpx.

$\rightsquigarrow \mathrm{Cyl}(f)$: chain cpx.

$\mathrm{Cyl}(f)_n = B_n \oplus B_{n-1} \oplus C_n$ &

$$\begin{bmatrix} d_B & \mathrm{id} & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{bmatrix} : \begin{matrix} B_n \\ \oplus \\ B_{n-1} \\ \oplus \\ C_n \end{matrix} \longrightarrow \begin{matrix} B_{n-1} \\ \oplus \\ B_{n-2} \\ \oplus \\ C_{n-1} \end{matrix}$$

$$\begin{bmatrix} d_B & \mathrm{id} & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{bmatrix} \begin{bmatrix} d_B & \mathrm{id} & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

§ Triangulated categories

K : cat. $T : K \longrightarrow K$ auto.

<u>Def</u> (1) A triangle on an ordered triple (A, B, C) of objs of K is a triple (u, v, w) of morphisms, where $u : A \to B$, $v : B \to C$ & $w : C \to TA$.

$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "v"'] \end{tikzcd}$$

or $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$.

(2) A morphism of triangles is a triple (f, g, h) forming a comm. diagram in K

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "v"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, "w"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & TA \arrow[d] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA' \end{tikzcd}$$

§ Def. of Triangulated categories

<u>Def</u> (Grothendieck, Verdier)

An additive category K is called a triangulated category if it is equipped with an automorphism $T: K \to K$ & with a distinguished family of triangles (u, v, w) (called the ext triangles in K) which are subject to the following axioms:

<u>TR1</u> Every morphism $f: A \to B$ can be embedded in an ext. triangle (u, v, w). If $A = B$ & $C = 0$, then the triangle $(\mathrm{id}_A, 0, 0)$ is ext. If (u, v, w) is a triangle on (A, B, C), isom. to an ext. triangle (u', v', w')

on (A', B', C'), then (u, v, w) is also ext.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\cong"'] & B \arrow[r, "v"] \arrow[d, "\cong"] & C \arrow[r, "w"] \arrow[d, "\cong"] & TA \arrow[d, "\cong"] \\ A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] & TA' \end{tikzcd}$$

<u>TR2</u> (Rotation) If (u, v, w) is an ext. triangle on (A, B, C), then both its "rotates" $(v, w, -Tu)$ & $(-T^{-1}w, u, v)$ are ext triangles on (B, C, TA) & $(T^{-1}C, A, B)$, resp.

<u>TR3</u> (Morphisms) Given two ext. triangles

$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "v"'] \end{tikzcd} \quad \& \quad \begin{tikzcd} & C' \arrow[dl, "w'"'] & \\ A' \arrow[rr, "u'"] & & B' \arrow[ul, "v'"'] \end{tikzcd}$$

with morphisms $f: A \to A'$, $g: B \to B'$
s.t. $gu = u'f$, $\exists$ a morphism $h: C \to C'$ s.t.
(f, g, h) is a morphism of triangles.

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, dashed, "\exists h"] & TA \arrow[d, "Tf"] \\
A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA'
\end{tikzcd}
$$

<u>TR4</u> (The octahedral axiom)

$$
\begin{tikzcd}
A \arrow[rr, "u"] \arrow[dr, "v \circ u"'] & & B \arrow[rr] \arrow[dl, "v"] & & C' \arrow[dl] \\
& C \arrow[dl] \arrow[r] & & B' \arrow[lll, bend left=10] & \\
A' & & & &
\end{tikzcd}
$$

$+ \ \partial = \delta f.$

ext

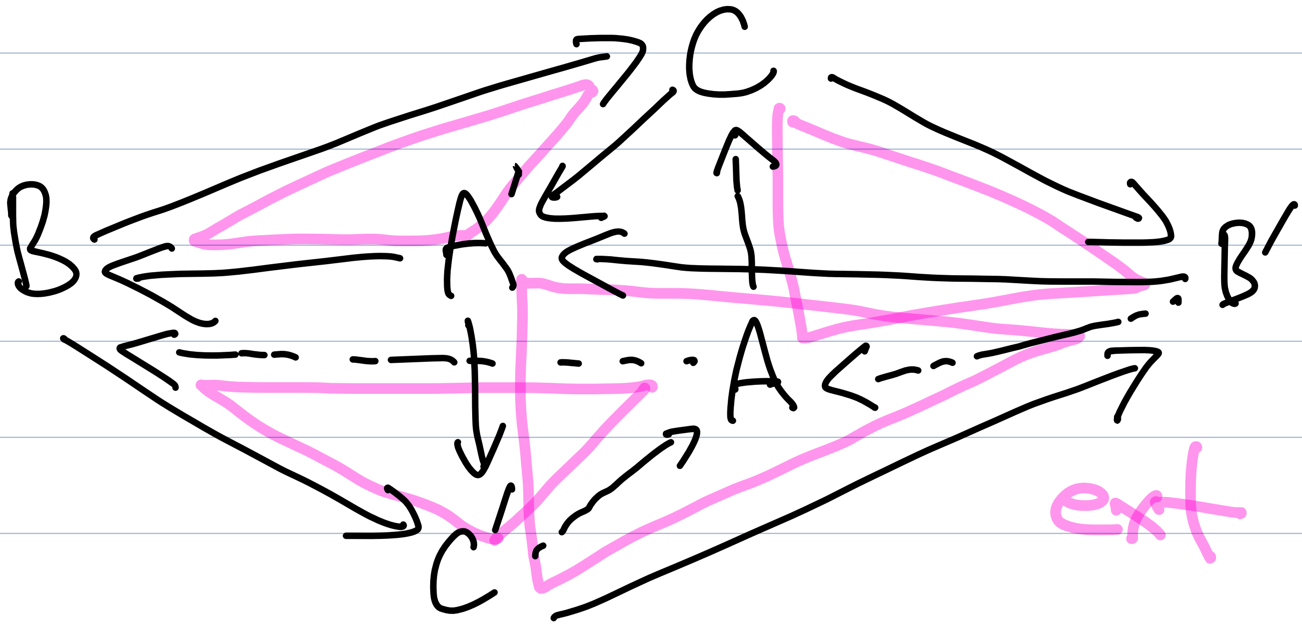
TR4 (The octahedral axiom)

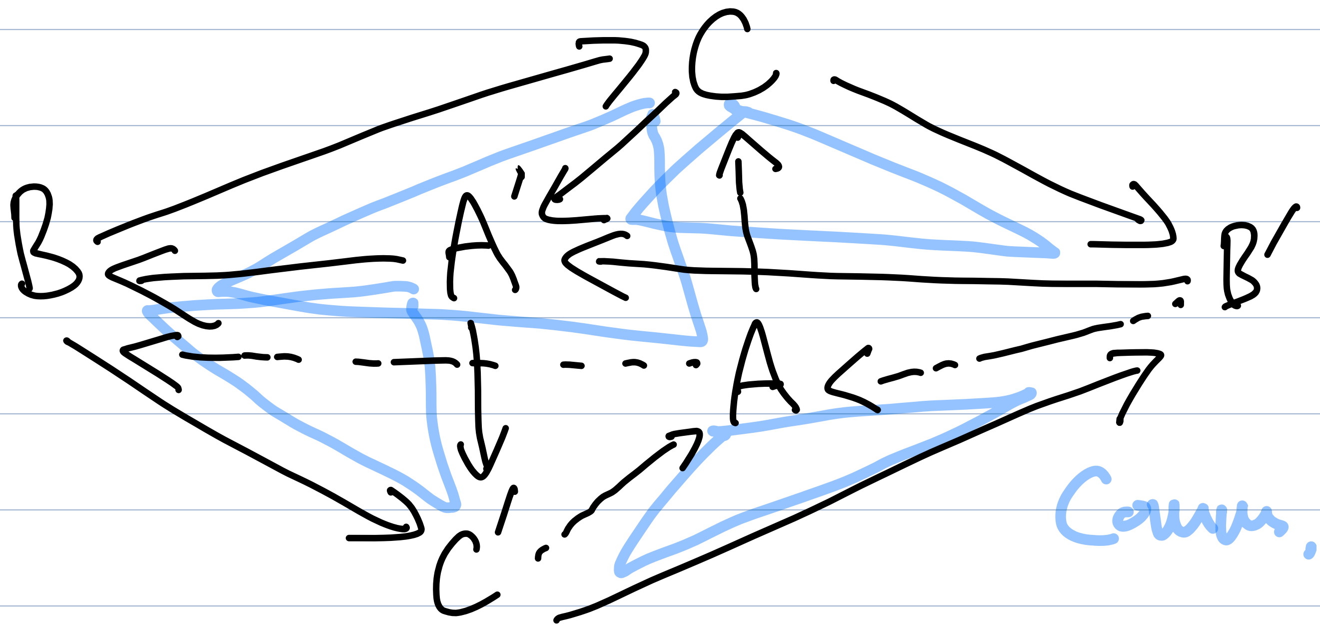
Given objs A, B, C, A', B', C' in $\mathcal{K}$, suppo-se there are 3 ext. triangles: (u, j, ∂) on (A, B, C'); (v, x, i) on (B, C, A'), (vu, y, δ) on (A, C, B'). Then there is 4th ext. triangle $(f, g, (Tj)i)$ on (C', B', A') s.t. in the following octahedron

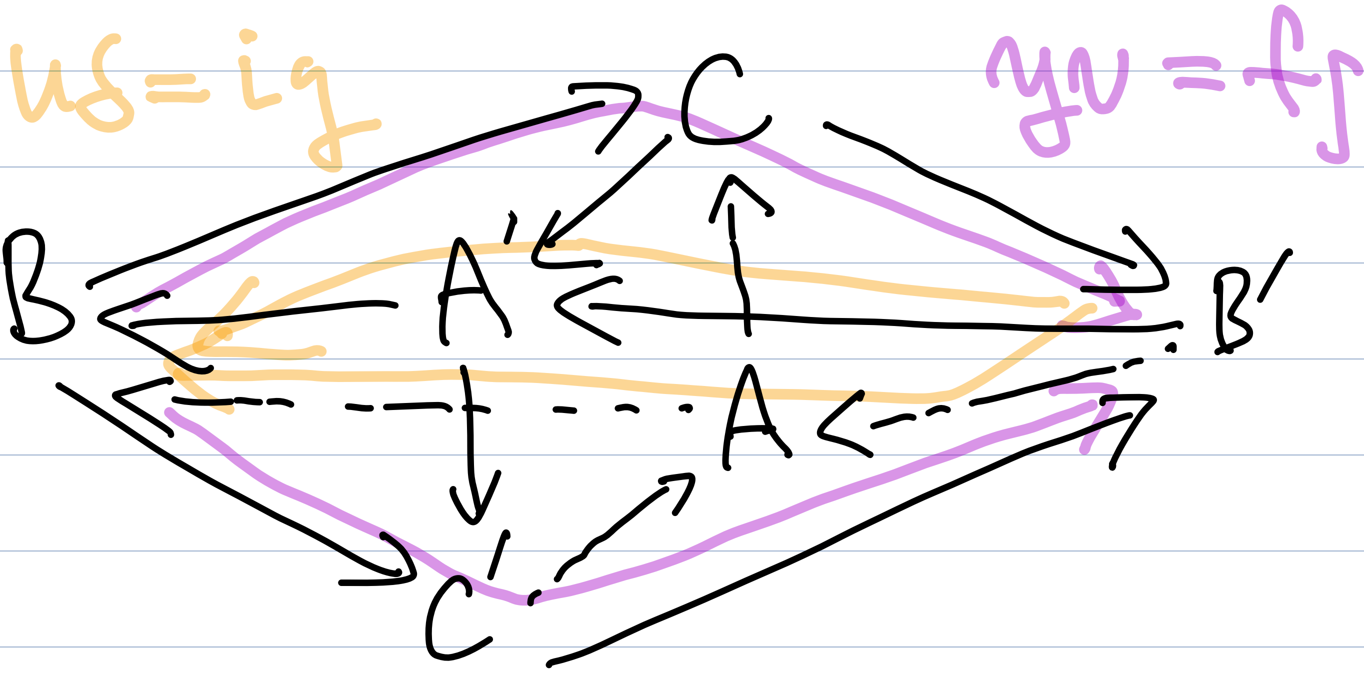
$$\begin{tikzcd}
& A' \arrow[dl, "(Tj)i"'] & \\
C' \arrow[rr, "f"'] & & B' \arrow[ul, "g"']
\end{tikzcd}$$

$$\begin{tikzcd}
& & C \arrow[dl, "x"'] \arrow[drr, "y"] & & \\
B \arrow[urr, "v"] \arrow[ddr, "j"'] & A' \arrow[l, "i"'] \arrow[dd, dashed, "(Tj)i"] & & & B' \arrow[lll, "\exists g"'] \arrow[dl, dashed, "\delta"'] \\
& & & A \arrow[uul, "vu"'] \arrow[ulll, dashed, "u"] & \\
& C' \arrow[urr, dashed, "\partial"] \arrow[uurrr, "\exists f"'] & & &
\end{tikzcd}$$

we have (1) the 4 ext triangles form four of the faces; (2) the remaining 4 faces comm. (that is, $\partial = \delta f : C' \to B' \to TA$ & $x = gy : C \to B' \to A'$); (3) $yv = fj : B \to B'$ & (4) $u\delta = ig : B' \to B$.


C
B
A'
B'
A
C'
ext


C
B
A'
B'
A
C'
Comm.


ug = iq
yv = fj
C
B
A'
B'
A
C'

Def (1) We say (u, v, w) is an ext. triangle on (A, B, C) if it is isomorphic to

$$A' \xrightarrow{u'} B' \xrightarrow{v'} \mathrm{Cone}(u') \xrightarrow{\delta} A'[-1]$$ coming in $K(\mathcal{A})$.

(2) $T : K(\mathcal{A}) \longrightarrow K(\mathcal{A})$

$$C_\cdot \longmapsto C_\cdot[-1]$$

Thm $K(\mathcal{A})$ is a triangulated cat.

TR1 $\quad id : C_\cdot \longrightarrow C_\cdot$

$\mathrm{Cone}(id)_n \; : \; C_{n-1} \oplus C_n$

$$\partial = \begin{bmatrix} -\partial & 0 \\ -id & \partial \end{bmatrix}$$

Claim $\mathrm{Cone}(id) = 0 \in K(\mathcal{A})$.

It is enough to show that $id_{\mathrm{Cone}(id)}$

$= 0.$

$$\begin{tikzcd}
\to & C_n \oplus C_{n+1} \arrow[r] \arrow[d] & C_{n-1} \oplus C_n \arrow[r] \arrow[d] & C_{n-2} \oplus C_{n-1} \arrow[r] \arrow[d] & {} \\
\to & C_n \oplus C_{n+1} \arrow[r] & C_{n-1} \oplus C_n \arrow[r] & C_{n-2} \oplus C_{n-1} \arrow[r] & {}
\end{tikzcd}$$

$(b, c) \longmapsto (-\partial b, \partial c - b)$

$(-c, 0)$

(b, c)

$= (b - \partial c, 0) +$

$(\partial c, c)$

$\Rightarrow$

$$\begin{tikzcd}
C \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"', "\cong"] & C \arrow[r] \arrow[d, "\cong"', "\mathrm{id}"] & 0 \arrow[r] \arrow[d, "\cong"] & C[-1] \arrow[d, "\cong"] \\
C \arrow[r, "\mathrm{id}"] & C \arrow[r] & \mathrm{Cone}(\mathrm{id}) \arrow[r] & C[-1]
\end{tikzcd}$$

$\square$ TR1.

TR2 $\qquad A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[-1]$

$$C = \mathrm{Cone}(u)$$

$$B \xrightarrow{v} \mathrm{Cone}(u) \longrightarrow \mathrm{Cone}(v)$$

$$B_n \xrightarrow{v} \mathrm{Cone}(u)_n \longrightarrow \mathrm{Cone}(v)_n$$

$$\mathrm{Cone}(u)_n = A_{n-1} \oplus B_n \qquad \mathrm{Cone}(v)_n = B_{n-1} \oplus A_{n-1} \oplus B_n$$

$$\mathrm{Cone}(v) \xrightarrow{\cong} A[-1] \in K(\mathcal{A})$$

$$B_{n-1} \oplus A_{n-1} \oplus B_n \longrightarrow A_{n-1}$$

$$\begin{pmatrix} \beta_1 \\ \alpha \\ \beta_2 \end{pmatrix} \longmapsto \alpha$$

$$B_{n-2}$$

$$\begin{bmatrix} -d & 0 & 0 \\ 0 & -d & 0 \\ 1 & -u & d \end{bmatrix} \qquad \begin{matrix} B_{n-2} \\ \oplus \\ A_{n-2} \\ \oplus \\ B_{n-1} \end{matrix} \longrightarrow A_{n-2}$$

$$-d\alpha$$

$$\begin{bmatrix} -d_A & 0 & 0 \\ 0 & -d & 0 \\ 1 & -u & d \end{bmatrix} \begin{pmatrix} \beta_1 \\ \alpha \\ \beta_2 \end{pmatrix} = \begin{pmatrix} -d\beta_1 \\ -d\alpha \\ \beta_1 - u\alpha + d\beta_2 \end{pmatrix}$$

$$A[-1] \longrightarrow \mathrm{Cone}(N)$$

$$A[-1]_n = A_{n-1} \longrightarrow \mathrm{Cone}(N)_n = \begin{matrix} B_{n-1} \\ \oplus \\ A_{n-1} \\ \oplus \\ B_n \end{matrix}$$

$$\alpha \longmapsto \begin{pmatrix} u\alpha \\ \alpha \\ 0 \end{pmatrix}$$

$$A_{n-2} \longrightarrow \mathrm{Cone}(N)_{n-1} = \begin{matrix} B_{n-2} \\ \oplus \\ A_{n-2} \\ \oplus \\ B_{n-1} \end{matrix}$$

$$-d\alpha \qquad \begin{bmatrix} -d_A & 0 & 0 \\ 0 & -d & 0 \\ 1 & -u & d \end{bmatrix} \begin{pmatrix} u\alpha \\ \alpha \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} u(-d\alpha) \\ \end{pmatrix} \qquad \begin{pmatrix} -d(u\alpha) \\ \end{pmatrix}$$

$$\begin{pmatrix} \cdots \\ -d\alpha \\ 0 \end{pmatrix} = \begin{pmatrix} \cdots \\ -d\alpha \\ \cancel{u\alpha - u\alpha} \end{pmatrix}$$

<u>Check</u> $A[-1] \to \mathrm{Cone}(N) \to A[-1] = \mathrm{id}_{A[-1]}$

$\mathrm{Cone}(N) \to A[-1] \to \mathrm{Cone}(N) \sim \mathrm{id}_{\mathrm{Cone}(N)}$.

$$\begin{pmatrix} \beta_1 \\ \alpha \\ \beta_2 \end{pmatrix} \longmapsto \alpha \longmapsto \begin{pmatrix} u\alpha \\ \alpha \\ 0 \end{pmatrix}$$

$$\begin{tikzcd}
 & B_{n-1} \oplus A_{n-1} \oplus B_n \arrow[r] \arrow[dl] \arrow[d] & B_{n-2} \oplus A_{n-2} \oplus B_{n-1} \arrow[dl] \\
B_n \oplus A_n \oplus B_{n+1} \arrow[r] & B_{n-1} \oplus A_{n-1} \oplus B_n &
\end{tikzcd}$$

$$\begin{pmatrix} \beta_1 \\ \alpha \\ \beta_2 \end{pmatrix} \longmapsto \begin{pmatrix} -d\beta_1 \\ -d\alpha \\ \beta_1 - u\alpha + d\beta_2 \end{pmatrix} \longmapsto \begin{pmatrix} \beta_1 - u\alpha + d\beta_2 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \beta_1 \\ \alpha \\ \beta_2 \end{pmatrix} \longmapsto \begin{pmatrix} \beta_2 \\ 0 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} -d\beta_2 \\ 0 \\ \beta_2 \end{pmatrix}$$

$$\begin{pmatrix} -d\beta_2 \\ 0 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \beta_1 - u\alpha + d\beta_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta_1 - u\alpha \\ 0 \\ \beta_2 \end{pmatrix}$$

$$\begin{pmatrix} \beta_1 \\ \alpha \\ \beta_2 \end{pmatrix} \longmapsto \begin{pmatrix} \beta_1 - u\alpha \\ 0 \\ \beta_2 \end{pmatrix}$$

Check the other conditions in similar ways!

TR3 We may suppose that $C = \mathrm{Cone}(u)$ & $C' = \mathrm{Cone}(u')$; the map h is given by the naturality of the mapping cone construction.

TR4 We may assume that the given triangles are strict, that is $C' = \mathrm{Cone}(u)$, $A' = \mathrm{Cone}(v)$ & $B' = \mathrm{Cone}(vu)$.

$$
\begin{tikzcd}
 & & C \arrow[dl, "x"'] \arrow[drr, "y"] & & \\
B \arrow[urr, "v"] \arrow[ddr, "j"'] & A' \arrow[l, "i"'] \arrow[dd, "(Tj)i"] & & & B' \arrow[lll, "\exists g"'] \arrow[dl, dashed, "\delta"'] \\
 & & & A \arrow[uul, "vu"'] \arrow[ulll, dashed, "u"] & \\
 & C' \arrow[urr, dashed, "\partial"] \arrow[uurrr, "\exists f"'] & & &
\end{tikzcd}
$$

Define $C'_{\cdot} \xrightarrow{f_{\cdot}} B'_{\cdot}$ by $f_n(a,b) = (a, v(b))$

$$C'_n = \begin{matrix} A_{n-1} \\ \oplus \\ B_n \end{matrix} \xrightarrow{\begin{pmatrix} id & 0 \\ 0 & v \end{pmatrix}} \begin{matrix} A_{n-1} \\ \oplus \\ C_n \end{matrix} = B'_n$$

& define $B'_\cdot \xrightarrow{g_\cdot} A'_\cdot$ by $g_n(a,c) = (u(a), c)$

$$B'_n = \begin{matrix} A_{n-1} \\ \oplus \\ C_n \end{matrix} \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & id \end{pmatrix}} \begin{matrix} B_{n-1} \\ \oplus \\ C_n \end{matrix} = A'_n$$

There are chain maps & $\partial = \delta f$ & $x = gy$.

$$\begin{tikzcd}
C'_n \arrow[rr, "(id,0)"] \arrow[dr, "f_n = \begin{pmatrix} id & 0 \\ 0 & N \end{pmatrix}"'] & & (A[-1])_n \\
& B'_n \arrow[ur, "(id,0) = \delta_n"'] &
\end{tikzcd}$$

$$C'_n = A_{n-1} \oplus B_n, \quad B'_n = A_{n-1} \oplus C_n, \quad (A[-1])_n = A_{n-1}$$

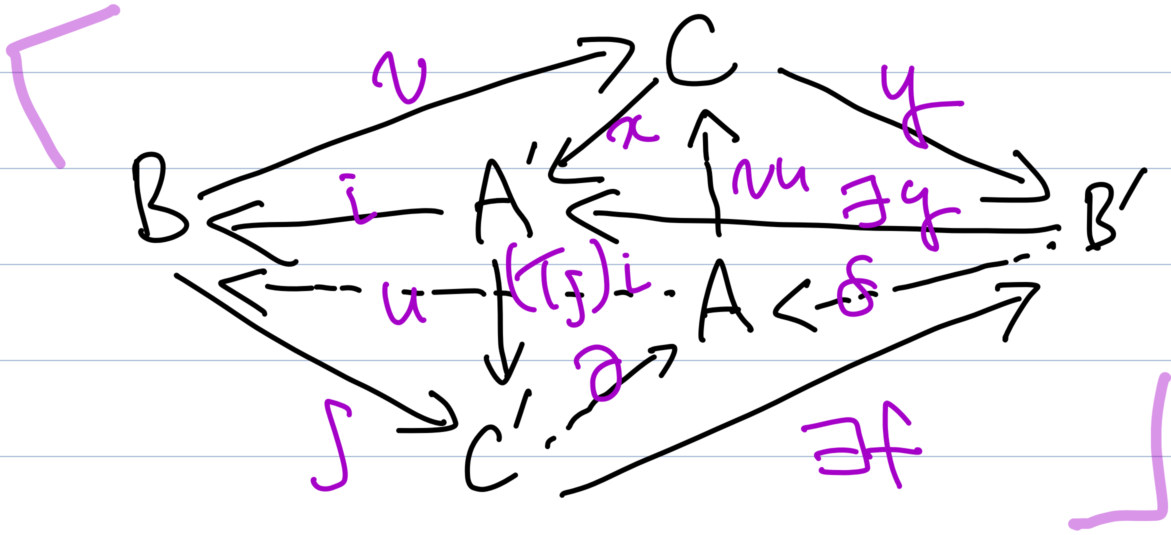
$$\left[(1,0)\begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix} = (1,0) \right]$$

$$\begin{tikzcd}
C_n \arrow[rr, "\begin{pmatrix} 0 \\ id \end{pmatrix} = x_n"] \arrow[dr, "y_n = \begin{pmatrix} 0 \\ id \end{pmatrix}"'] & & A'_n \\
& B' \arrow[ur, "g_n = \begin{pmatrix} u & 0 \\ 0 & id \end{pmatrix}"'] &
\end{tikzcd}$$

$$A'_n = B_{n-1} \oplus C_n, \quad B' = A_{n-1} \oplus \cdots$$

$$\left[\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$\oplus$
C_n



Since the deg n part of $\mathrm{cone}(f)$ is $C'_{n-1} \oplus B'_n = A_{n-2} \oplus B_{n-1} \oplus A_{n-1} \oplus C_n$, $\exists$ a natural inclusion γ of A' into $\mathrm{cone}(f)$ s.t. the following diagram of chain cpxes comms.

$$\begin{tikzcd}
C' \arrow[r, "f"] \arrow[d, equal] & B' \arrow[r, "g"] \arrow[d, equal] & A' \arrow[r, "(T_f)i"] \arrow[d, "j\gamma"] & C'[-1] \arrow[d, equal] \\
C' \arrow[r, "f"] & B' \arrow[r] & \mathrm{cone}(f) \arrow[r] & C'[-1].
\end{tikzcd}$$

Define $\varphi : \mathrm{cone}(f) \longrightarrow A'$ by

$$\begin{array}{ccc} \varphi_n = \mathrm{cone}(f)_n & \longrightarrow & A' \\ \| & & \| \\ A_{n-2} \oplus B_{n-1} \oplus A_{n-1} \oplus C_n & & B_{n-1} \oplus C_n \\ (a_{n-2}, b, a_{n-1}, c) & \longmapsto & (b + u(a_{n-1}), c) \end{array}$$

<u>Check</u> $\varphi\gamma = \mathrm{id}_{A'}$ & $\gamma\varphi \sim_{\text{htpy}} \mathrm{id}_{\mathrm{cone}(f)}$

$\Rightarrow (f, g, (T_j)i)$ is an ext. triangle.

Other parts follow from the construction. $\square$